Section 3.5 Implicit Differentiation

Example: \( x^3 + y^3 = 6xy \). Find \( \frac{dy}{dx} \) and write the equation of the tangent line at \((3,3)\).

Solution: \( 3x^2 + 3y^2 \cdot \frac{dy}{dx} = 6y + 6x \cdot \frac{dy}{dx} \)

\[
\Rightarrow \quad \frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2
\]

\[
\Rightarrow \quad \frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}
\]

Slope of the tangent line at \((3,3)\): \( \frac{dy}{dx}(3,3) = \frac{2(3) - 3^2}{3^2 - 2(3)} = \frac{6 - 9}{9 - 6} = \frac{-3}{3} = -1 \)

Equation: \((y - 3) = -1(x - 3)\)

\( \Rightarrow \quad y - 3 = -x + 3 \)

\( \Rightarrow \quad y + x = 6 \)
Derivatives of Inverse Trigonometric Functions

\[ y = \sin^{-1}x \text{ means } \sin y = x \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \]

\[ \sin y = x \]
\[ \frac{d}{dx}(\sin y) = \frac{d}{dx}(x) \]
\[ \cos y \cdot \frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos y} \]
\[ \cos y \text{ is positive} \]
\[ \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}} \]

\[ (\sin^{-1}x)^1 = \frac{1}{\sqrt{1 - x^2}} \]

\[ y = \tan^{-1}x \quad \Leftrightarrow \quad \tan y = x \quad \frac{-\pi}{2} < y < \frac{\pi}{2} \]
\[ \tan y = x \]
\[ \Rightarrow \quad \frac{d}{dx}(\tan y) = \frac{d}{dx}(x) \]
\[ \sec^2 y \cdot \frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} \]
\[ \Rightarrow \quad \frac{dy}{dx} = \frac{1}{1 + x^2} \]
\[ (\tan^{-1}x)' = \frac{1}{1+x^2} \]

**Derivatives of inverse trigonometric functions.**

- \[ \frac{d}{dx} (\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \]
- \[ \frac{d}{dx} (\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \]
- \[ \frac{d}{dx} (\tan^{-1}x) = \frac{1}{1+x^2} \]
- \[ \frac{d}{dx} (\cot^{-1}x) = -\frac{1}{1+x^2} \]
- \[ \frac{d}{dx} (\sec^{-1}x) = \frac{1}{x \sqrt{x^2-1}} \]
- \[ \frac{d}{dx} (\csc^{-1}x) = -\frac{1}{x \sqrt{x^2-1}} \]

3.6 **Derivatives of Logarithmic Functions**

We will show that \[ \frac{d}{dx} (\log_a x) = \frac{1}{x \cdot \ln a} \]

**Proof**

\[ \log_a x = y \iff a^y = x \]

\[ \frac{d}{dx} (a^y) = \frac{d}{dx} (x) \]

\[ a^y \cdot \ln a \cdot \frac{dy}{dx} = 1 \]

\[ \Rightarrow \frac{dy}{dx} = \frac{1}{a^y \cdot \ln a} = \frac{1}{x \cdot \ln a} \]
Put \( q = e \)

\[ \Rightarrow \frac{d}{dx} (\ln x) = \frac{1}{x \cdot \ln e} \]

(2)

\[ \Rightarrow \frac{d}{dx} (\ln x) = \frac{1}{x} \]

Example

\[ y = \ln (x^5 + 3) \]

\[ f(x) = x^5 + 3 \quad \Rightarrow \text{inner function} \quad f'(x) = 5x^4 \]

\[ g(x) = \ln x \quad \Rightarrow \text{outer function} \quad g'(x) = \frac{1}{x} \]

\[ y' = \frac{1}{x^5 + 3} \cdot 5x^4 \]

Note: We can combine Formula 2 and the Chain Rule

(3)

\[ \frac{d}{dx} (\ln (f(x))) = \frac{1}{f(x)} \cdot f'(x) \]

Example

\[ y = \ln (\tan x) \quad y' = ? \]

Solution

By applying formula 3

\[ y' = \frac{1}{\tan x} (\tan x)' = \frac{1}{\tan x} \cdot \sec^2 x \]
Example: \[ y = \ln \left( \frac{x^2 + 1}{3x + 5} \right) \Rightarrow y' = ? \]

Solution: \[ y = \ln (x^2 + 1) - \ln (3x + 5) \]
\[ y' = \frac{1}{x^2 + 1} \cdot 2x - \frac{1}{3x + 5} \cdot 3 \]

Example: Find \( f'(x) \) if \( f(x) = \ln |x| \)

Solution: \[ f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases} \]
\[ \Rightarrow f'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ -\frac{1}{-x} = \frac{1}{x} & \text{if } x < 0 \end{cases} \]
\[ \Rightarrow f'(x) = \frac{1}{x} \quad \text{for all } x \neq 0 \]

\[ \Rightarrow \text{It is worth remembering:} \]

\[ \frac{d}{dx} \ln |x| = \frac{1}{x} \]
Section 3.5 Implicit Differentiation

Example \( x^3 + y^3 = 6xy \). Find \( \frac{dy}{dx} \) and write the equation of the tangent line at (3,3).

Solution \( 3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6x \). \( \frac{dy}{dx} \)

\[
\frac{dy}{dx} (3y^2 - 6x) = 6y - 3x^2
\]

\[
\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}
\]

Slope of the tangent line at (3,3)

\[
\frac{dy}{dx} (3,3) = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = \frac{6 - 9}{9 - 6} = -\frac{3}{1}
\]
Logarithmic Differentiation:

The calculation of derivatives of complicated functions can often be simplified by taking logarithms.

Example: \( y = x^{(\sqrt{x})} \) \( \Rightarrow \) \( y' = \) ?

\[
\ln y = \ln (x^{\sqrt{x}})
\]

\[
\Rightarrow \ln y = \sqrt{x} \cdot \ln x
\]

\[
\Rightarrow (\ln y)' = \left(\frac{1}{2\sqrt{x}} \cdot \ln x + \sqrt{x} \cdot \frac{1}{x}\right)
\]

\[
\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \cdot \ln x + \sqrt{x} \cdot \frac{1}{x}
\]

\[
\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{\ln x}{2\sqrt{x}} + \frac{1}{x}
\]

\[
\Rightarrow \frac{dy}{dx} = y \left( \frac{\ln x + 2}{2\sqrt{x}} \right)
\]

\[
\Rightarrow \frac{dy}{dx} = x^{\sqrt{x}} \left( \frac{\ln x + 2}{2\sqrt{x}} \right)
\]
Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation \( y = f(x) \) and use the Laws of Logarithms to simplify
2. Differentiate implicitly with respect to \( x \)
3. Solve the resulting equation for \( y' \).

\[
\begin{align*}
y = a^x & \quad \Rightarrow \quad y' = ? \\
y = a^x & \Rightarrow \ln y = \ln a^x \\
& \Rightarrow \quad \ln y = x \cdot \ln a
\end{align*}
\]

\[ (a^x)' = a^x \cdot \ln a \]
General Rule (Use previous example and combine it with Chain Rule)

\[
\frac{d}{dx} (a^{f(x)}) = a^{f(x)} \ln a \cdot f'(x)
\]

Example

\[y = a^{x^2 + 6x} \Rightarrow y' = ?\]

\[y' = a^{x^2 + 6x} \ln a \cdot (2x + 6)\]

The Number e as a Limit

\[f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}\]

\[f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\ln (1+h) - \ln 1}{h}\]

\[= \lim_{h \to 0} \frac{\ln (1+h)}{h}\]

\[= \lim_{h \to 0} \frac{1}{h} \cdot \ln (1+h)\]

\[= \lim_{h \to 0} \frac{\ln (1+h)}{1/h}\]
\[ f'(1) = 1 \]

\[ \Rightarrow \lim_{{x \to 0}} \ln (1+x)^{\frac{1}{x}} = 1 \]

We know that \( e^x \) is continuous on everywhere.

\[ e = e^1 = e^{\lim_{{x \to 0}} \ln (1+x)^{\frac{1}{x}}} = \lim_{{x \to 0}} e^{\ln (1+x)^{\frac{1}{x}}} = \lim_{{x \to 0}} (1+x)^{\frac{1}{x}} \]

\[ \Rightarrow \boxed{e = \lim_{{x \to 0}} (1+x)^{\frac{1}{x}}} \]

Put \( n = \frac{1}{x} \), as \( x \to 0^+ \), \( n \to \infty \)

So an alternative version of \( \boxed{15} \)

\[ \boxed{6} \]

\[ e = \lim_{{n \to \infty}} \left( 1 + \frac{1}{n} \right)^n \]
Ex: \( y = (\cos x)^x \Rightarrow y' = ? \)

Solution:
\[
\begin{align*}
\ln y &= x \ln(\cos x) \\
\frac{1}{y} \frac{dy}{dx} &= x \frac{\ln(\cos x)}{\cos x} - \sin x \\
\frac{1}{y} \frac{dy}{dx} &= \ln(\cos x) - x \frac{\sin x}{\cos x} \\
\frac{dy}{dx} &= y \left( \ln(\cos x) - x \tan x \right) \\
y' &= (\cos x)^x \left( \ln(\cos x) - x \tan x \right)
\end{align*}
\]

Ex: \( y = \ln \left( x + \sqrt{x^2 + 3} \right) \Rightarrow y' = ? \)

Solution:
\[
\begin{align*}
y' &= \frac{1}{x + \sqrt{x^2 + 3}} \left( x + \sqrt{x^2 + 3} \right)' \\
y' &= \frac{1}{x + \sqrt{x^2 + 3}} \left( 1 + \frac{1}{2 \sqrt{x^2 + 3}} \cdot 2x \right)
\end{align*}
\]
3.7 Rates of Change in the Natural and Social Sciences

\[
\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \rightarrow \text{average rate of change of } y \text{ with respect to } x \text{ over the interval } [x_1, x_2]
\]

We can interpret it as "slope of the secant line" $PQ$

$P = (x_1, f(x_1))$

$Q = (x_2, f(x_2))$

Its limit as $\Delta x \to 0$ is derivative $f'(x_1)$, which can therefore be interpreted as the \textbf{instantaneous} rate of change of $y$ with respect to $x$ \textbf{OR} slope of the tangent line at $P(x_1, f(x_1))$.

\[\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\]

We now look at some interpretations in the natural and social sciences.
If \( s=f(t) \) is the position function,

\[
\frac{\Delta s}{\Delta t} \rightarrow \text{average velocity over a time period } \Delta t
\]

\[ V = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \quad \text{"instantaneous velocity"} \]

\[ V'(t) = s'(t) = a(t) \quad \text{"acceleration"} \]

**Ex: The position of a particle is given by the equation**

\[ s = f(t) = t^3 - 6t^2 + 9, t \]

where \( t \) is measured in seconds and \( s \) in meters.

(a) Find velocity and acceleration function.

(b) When is the particle at rest?

(c) When is the particle moving forward?

So:

\[ V(t) = 3t^2 - 12t + 9 \]

\[ \text{Acceleration } (t) = 6t - 12 \]

(b) The particle is at rest when \( V(t) = 0 \)

\[ 3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t - 1)(t - 3) = 0 \]

\[ t = 1 \quad \text{&} \quad t = 3 \]
(c) The particle moves in the positive direction when 
\[ v(t) > 0, \text{ that is } \]
\[ 3t^2 - 12t + 9 > 0 \]
\[ 3(t-1)(t-3) > 0 \]

Thus the particle moves in positive direction when \( t < 1 \) and \( t > 3 \).

It moves in the negative direction when \( 1 < t < 3 \).

\[ \text{Biology} \]

Let \( n = f(t) \) be the number of individuals in an animal or plant population at time \( t \).

Change in population size between \( t = t_1 \) and \( t = t_2 \)

\[ \Delta n = f(t_2) - f(t_1) \]

Average rate of growth = \[ \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1} \]
Instantaneous rate of growth: \[ \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt} \]

Growth rate: \[ \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt} \]

Example: The population function for E. coli bacteria is \( n = n_0 \cdot 2^t \) (no is starting population).

Rate of growth of the bacteria is \( \frac{dn}{dt} = \frac{dn}{dt}(n_0 \cdot 2^t) \)

\[ = n_0 \cdot \frac{d}{dt}(2^t) \]

\[ = n_0 \cdot 2^t \cdot \ln 2 \]

\( \Rightarrow \) Growth rate is \( n_0 \cdot 2^t \cdot \ln 2 \)

\( \Rightarrow \) After 4 hours, the bacteria population is growing at a rate of about \( 11.09 \) bacteria per hour.

(assuming \( n_0 = 100 \))
Economics

Suppose \( C(x) \) is the total cost that a company incurs in producing \( x \) units of a certain commodity. 

\( C \) is called "cost function".

If the number of items produced is increased from \( x_1 \) to \( x_2 \), then the additional cost \( \Delta C = C(x_2) - C(x_1) \).

Average rate of change of the cost:

\[
\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}
\]

Take \( \Delta x \to 0 \)

Instantaneous rate of change of the cost with respect to the number of items produced, is called the "marginal cost" by economists.

\[
\text{Marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}
\]
Suppose a company has estimated the cost (in dollars) of producing \( x \) items is

\[
C(x) = 10.000 + 5x + 0.01x^2
\]

**Marginal Cost Function**

\[
C'(x) = 5 + 0.022x
\]

Marginal cost when the production level is 500 items

\[
\Rightarrow C'(500) = $15 / \text{item}
\]

This gives the rate at which costs are increasing with respect to the production level when \( x = 500 \) and predicts the cost of 501st item.

Notice that \[
C'(500) \approx C(501) - C(500)
\]